

# On chaotic behavior of gravitating stellar shells.

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Motion of two gravitating spherical stellar shells around a massive central body is considered. Each shell consists of point particles with the same specific angular momenta and energies. In the case when one can neglect the influence of gravitation of one ("light") shell onto another ("heavy") shell ("restricted problem") the structure of the phase space is described. The scaling laws for the measure of the domain of chaotic motion and for the minimal energy of the light shell sufficient for its escape to infinity are obtained.

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**Chaotic motion of a system can result in crossing a border in the phase space separating the area where disintegration of the system is possible. In the present paper this phenomenon is studied in a system of two gravitating stellar shells oscillating around a massive central body. Similar phenomena take place in planetary and molecular dynamics.**

## I. INTRODUCTION

Dynamical processes around supermassive black holes (SBH) in quasars, blazars and active galactic nuclei are characterized by violent phenomena, leading to formation of jets and other outbursts. Here we consider a model problem for a shell outburst from a SBH surrounded by a dense massive stellar cluster due to a pure gravitational interaction of shells oscillating around SBH.

Investigation of spherical stellar clusters using shell approximation was started by Hénon [9]. In this approximation a stellar cluster is considered as a collection of spherical stellar shells. Each shell consists of stars with the same specific angular momenta and energies. In the process of motion the angular momenta are conserved while energies are changing, yet taking the same values for all the

stars in one shell. The shell itself stay spherical. Shell approximation has been successfully applied for investigation of stability [10], violent relaxation and collapse [8, 9, 15], leading to formation of a stationary cluster. Investigation of the evolution of a spherical stellar cluster taking into account various physical processes was done on the base of the shell model in the classical series of papers of L. Spitzer and his coauthors [21, 22, 23, 24, 25, 26, 27, 28], see also [19].

Numerical calculations of a collapse of stellar clusters in the shell approximation [6, 7, 29] have shown, that even if all shells are initially gravitationally bound, after a number of passages through each other some shells obtain sufficient energy to be thrown to infinity.

Here we consider a simplified problem of motion of two massive spherical shells, each consisting of stars with the same specific angular momentum and energy, around SBH.

Development of chaos during the motion of two gravitating intersecting shells was found first by Miller and Youngkins [17] in an oversimplified case of purely radial motion and reflecting inner boundary. In [4, 5] chaotic behavior was found in a more realistic model considering stars with the same specific angular momentum and energy dispersed isotropically over the spherical shell, and each star moving along its ballistic trajectory in the gravitational field of the central body and shells. In the current paper we consider the same model mainly in the case when one can neglect the influence of one ("light") shell onto the motion of another ("heavy") shell. In this simplified case we describe a structure of the phase space and obtain the scaling laws for the measure of the domain of chaotic

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motion and for the minimal energy of the light shell sufficient for its escape to infinity.

We follow the approach which was used in [20] for analysis of chaotic dynamics and escape to infinity in the planar restricted circular three body problem. This approach consists of constructing asymptotic formulas for Poincaré return map and identification of regions of regular and chaotic dynamics in the phase space of this map. Region of regular dynamics is the region where Kolmogorov-Arnold-Moser (KAM) theory (see, e.g. [1, 2]) guarantees existence of many invariant curves of the map. These curves are barriers preventing the escape to infinity. Region of chaotic motion is the region where the phase expansion criterion (see, e.g. [30]) is satisfied. In this region chaotic diffusion allows the "light" shell to gain positive energy and then escape to infinity. One special property of the system under consideration is that for an open set of the problem's parameters, in the phase space there is a region where the Poincaré return map is continuous but not smooth (it has a singularity of the square root kind). In this region neither KAM-theory nor phase expansion criterion are applicable. Similar situation, though with a map that is discontinuous in a certain region in the phase space occur in the problem of motion of three gravitating parallel impenetrable sheets [14]. For general analysis of maps with discontinuities see [3] and references there.

The dynamics of a system of many gravitating spherical shells was studied from the viewpoint of statistical mechanics in [11, 12, 13] where, however, the escape to infinity was not studied because a bounded system was considered. An extensive study of escape to infinity in the rectilinear three body problem was performed in [16].

## II. A RESTRICTED PROBLEM.

Consider the motion of two stellar shells with masses  $m_1$  and  $m_2$  around the central body of mass  $M$ . Let  $J_1$  and  $J_2$  be specific angular momenta of shells. Consider the case when  $m_2 \ll m_1$ , so that one can neglect the influence of the second shell onto the motion of the first shell. We call this problem a restricted problem by the analogy with the restricted three body problem in celestial mechanics (the latter problem describes the motion of an asteroid under the action of Sun and Jupiter).

In what follows the first shell will be called the heavy one, and the second shell will be called the light one.

In the restricted problem, motion of the heavy shell is described by Hamiltonian function

$$H_1 = \frac{p_1^2}{2} + V_1, \quad V_1 = -\frac{G(M + m_1/2)}{r_1} + \frac{J_1^2}{2r_1^2}. \quad (1)$$

Here  $p_1$  is the momentum canonically conjugated to the shell radius  $r_1$ . Therefore, the change of  $r_1$  in time is the same as in the Kepler problem for a satellite with mass 1 and angular momentum  $J_1$  and a central body of mass  $(M + m_1/2)$ . For  $H_1 = h_1 < 0$  the behavior of  $r_1$  is described by usual formulas of the elliptic Keplerian motion:  $r_1 = r_1(l_1, h_1, J_1)$ ; here  $l_1$  is the mean anomaly of the satellite, i.e. the angular variable for which

$$\dot{l}_1 = \frac{\sqrt{\mu_1}}{a_1^{3/2}}, \quad \mu_1 = G(M + m_1/2), \quad a_1 = -\frac{\mu_1}{2h_1}. \quad (2)$$

Motion of the light shell is described by Hamiltonian

$$H_2 = \frac{p_2^2}{2} + V_2, \quad V_2 = \begin{cases} V_0, & \text{if } r_2 \leq r_1, \\ V_0 - Gm_1/r_2, & \text{if } r_2 \geq r_1, \end{cases} \quad (3)$$

where

$$V_0 = -\frac{GM}{r_2} + \frac{J_2^2}{2r_2^2}.$$

Here  $p_2$  is the momentum canonically conjugated to radius  $r_2$  of the light shell. In the Hamiltonian  $H_2$  one should consider  $r_1$  as a given function of time. Therefore, in the restricted problem motion of the light shell is described by a Hamiltonian system with one and a half degrees of freedom (one degree of freedom plus explicit dependence of the Hamiltonian on time). This model applies just as well to the case where the light massless shell is replaced by a massless point.

## III. POINCARÉ RETURN MAP FOR THE RESTRICTED PROBLEM.

Consider the motion of the light shell. At the time moments  $t$ , when  $r_2$  has a local minima on the trajectory, we mark on the cylinder  $\Phi = \{(h, \phi) : -\infty < h < \infty, \phi \bmod 2\pi\}$  the values of the light shell Hamiltonian  $H_2$  and of the heavy shell mean anomaly  $l_1$ :  $(H_2, l_1) \equiv (h, \phi)$ . Thus we obtain on  $\Phi$  a sequence of points  $\dots, (h^s, \phi^s), \dots, (h^{-1}, \phi^{-1}), (h^0, \phi^0), (h^1, \phi^1), \dots, (h^k, \phi^k), \dots$

This sequence terminates at the right if the shell escapes to infinity. It can be terminated at the left as well, but the terms of this sequence with numbers smaller than certain negative number do not have physical meaning. The points of the constructed sequence are mapped to each other under the action of Poincaré return map defined as follows. Consider the motion of the light shell that starts at time moment  $t = \tilde{t}$  with  $p_2 = 0$ . Denote initial values of  $H_2$  and  $l_1$  as  $h$  and  $\phi$ . The initial value of  $r_2$  satisfies the equation  $H_2 = h$ . Let this initial value  $r_2$  be the smallest of two roots of equation  $H_2 = h$ . Therefore, at the time moment  $\tilde{t}$  the value of  $r_2$  on the trajectory has a local minimum. Denote  $t'$  as the first time moment after  $\tilde{t}$  such that at this time moment again  $p_2 = 0$  and  $r_2$  on the trajectory has a minimum, provided that such a time moment exists. Denote as  $h'$  and  $\phi'$  values of  $H_2$  and  $l_1$  corresponding to  $t = t'$ . By definition, Poincaré return map  $\Pi$  is given by the formula  $\Pi(h, \phi) = (h', \phi')$ . Then  $(h^{(k+1)}, \phi^{(k+1)}) = \Pi(h^{(k)}, \phi^{(k)})$ . The map  $\Pi$  is defined on the part of the cylinder  $\Phi$  where values of  $(h, \phi)$  allow the shell to make at least one complete oscillation. The map  $\Pi$  preserves the area  $dhd\phi$  on  $\Phi$ ; this follows from preservation of the phase volume by a Hamiltonian system (see, e.g. Refs. [1, 2]).

Together with map  $\Pi$  we will use modified return map  $\hat{\Pi}$  defined as follows. Let  $\tilde{t}, t', \phi, \phi'$  be the same as before. Denote by  $\hat{h}$  the value of  $H_2$  at the last preceding to  $\tilde{t}$  time moment when  $r_2$  has a maximum. Denote by  $\hat{h}'$  the value of  $H_2$  at the first time moment after  $\tilde{t}$  when  $r_2$  has a maximum. Define map  $\hat{\Pi}$  by the formula  $\hat{\Pi}(\hat{h}, \phi) = (\hat{h}', \phi')$ . The map  $\hat{\Pi}$  is defined on the part of the cylinder  $\hat{\Phi} = \{(\hat{h}, \phi) : -\infty < \hat{h} < \infty, \phi \bmod 2\pi\}$  where  $\hat{h} < 0$ . The map  $\hat{\Pi}$  preserves the area  $(\partial h / \partial \hat{h})_\phi d\hat{h}d\phi$  on  $\hat{\Phi}$ . For a given  $(\hat{h}^l, \phi^l)$  the map  $\hat{\Pi}$  generates on  $\hat{\Phi}$  the sequence of points  $\dots, (\hat{h}^s, \phi^s), \dots, (\hat{h}^{-1}, \phi^{-1}), (\hat{h}^0, \phi^0), (\hat{h}^1, \phi^1), \dots, (\hat{h}^k, \phi^k), \dots$  such that  $(\hat{h}^{(k+1)}, \phi^{(k+1)}) = \hat{\Pi}(\hat{h}^{(k)}, \phi^{(k)})$ . At small values of  $\hat{h}$  the time interval between  $\tilde{t}$  and  $t'$  is determined in the main approximation by the value  $\hat{h}'$ :  $t' - \tilde{t} = 2\pi\mu_2 / (-2\hat{h}')^{3/2}$ ,  $\mu_2 = G(M + m_1)$ .

This is the reason why for small  $\hat{h}$  map  $\hat{\Pi}$  is more convenient for the analysis than  $\Pi$ . In what follows we omit for brevity the symbol " over  $\hat{h}, \hat{\Phi}$ .

In what follows we consider the case when the heavy shell's mass is much smaller than the mass of the central body:  $m_1 = \varepsilon M$ ,  $0 < \varepsilon \ll 1$ . In this case map  $\hat{\Pi}$  has the form

$$h' = h + \varepsilon f(h, \phi, \varepsilon), \quad (4)$$

$$\phi' = \phi + 2\pi \frac{\mu_2}{\mu_1} \left( \frac{h_1}{h'} \right)^{3/2} + \varepsilon g(h', \phi', \varepsilon). \quad (5)$$

Functions  $f, g$  can be extended as continuous functions up to the argument value  $h = 0$ . The value  $\varepsilon f(h, \phi, \varepsilon)$  is the sum of jumps of the light shell's potential energy due to passage of shells through each other between two consecutive time moments when the light shell radius takes maximal values. To find of  $\varepsilon f(h, \phi, 0)$ , one should calculate the sum of the energy jumps under the assumption that motion of the light shell is not affected by the heavy shell (i.e. this motion is Keplerian one with total energy  $h$ ).

Depending on the properties of the shell motion at  $\varepsilon = 0$ , one can define three main types of  $h$  values.

#### 1. Type A.

At all initial phases  $\phi$  of the heavy shell motion, the light shell on the period of its motion passes through the heavy shell at non-zero relative velocity. (If  $h = 0$ , i.e. motion of the shell stars is parabolic, then in this and following definitions one should consider as a period of the light shell motion the whole infinite time interval between arrival of the shell from infinity and its departure to infinity.) In this case during a period of the light shell motion the number of passages of shells through each other is the same for any initial phase of the heavy shell motion. Therefore, the number of terms in the sum for  $f(h, \phi, 0)$  is the same for any  $\phi$ . At small enough  $\varepsilon$ , the sum for  $f(h, \phi, \varepsilon)$  has the same number of the terms. Therefore, function  $f(h, \phi, \varepsilon)$  is analytic in  $(h, \phi)$ , and it can be continued analytically in some neighborhood of a circle around  $h = 0$ .

#### 2. Type B.

There exists certain initial phase  $\phi_*$  of the heavy shell motion such that at some time moment the shells superimpose with zero relative velocity and

non-zero relative acceleration, and absolute velocities of both shells at this moment are different from zero. At this moment trajectories of the shells have quadratic tangency. When  $\phi$  deviates from  $\phi_*$  in one direction, the superimposing of the shells disappears. When  $\phi$  deviates from  $\phi_*$  in another direction, this superimposing decomposes into two close shell crossings (say, light shell surpasses the heavy one and then almost immediately the heavy shell surpasses the light one). In this latter case two new terms appear in the sum for  $f(h, \phi, 0)$ ; these terms are close by absolute values, and they have different signs. The sum of these terms is of order  $\sqrt{|\phi - \phi_*|}$ . Therefore, in this case function  $f(h, \phi, 0)$  has at  $\phi = \phi_*$  the singularity of type  $\sqrt{\phi - \phi_*}$  or  $\sqrt{\phi_* - \phi}$ . At small variations of  $h$  this singularity persists. Function  $f(h, \phi, \varepsilon)$  has the same singularity at small  $\varepsilon$ . Value  $r_2$  that corresponds to time moment of tangency of shell trajectories is uniquely defined. Indeed, at the moment of tangency there should be  $r_1 = r_2$ ,  $h_1 + J_1^2/2r_1^2 = h + J_2^2/2r_2^2$  (because velocities of the shells are equal to each other) and  $J_1^2/2r_1^3 \neq J_2^2/2r_2^3$ . These conditions define a unique value of  $r_2$ .

### 3. Type C.

For any initial phase  $\phi$  of the heavy shell motion the shells never superimpose. Then at small enough  $\varepsilon$  motion of the light shell is also Keplerian.

There is also exceptional intermediate type of  $h$  for which the shells can meet with zero velocities (i.e. one of two extremal values of radius of one shell coincide with an extremal value of radius of another shell), but with nonzero relative acceleration. Another exceptional case is  $h = h_1$  and  $J_1 = J_2$ . In this case for some initial phase  $\phi$  of the heavy shell the radii of two shells coincide all the time.

## IV. DYNAMICS OF SHELLS IN THE RESTRICTED PROBLEM.

Let us consider first the case when the value  $h = 0$  is of type A. Consider in cylinder  $\Phi$  the annulus  $\alpha \leq h \leq 0$  such that all values  $h$  on this annulus are of the type A (here  $\alpha$  is a constant,  $\alpha < 0$  does not depend on  $\varepsilon$ ). In this annulus the map  $\hat{\Pi}$  is analogous to "Kepler map", which was introduced in [20]. Let us repeat for map  $\hat{\Pi}$  the

analysis from [20] (see also [18]). The trajectories of several initial points under the action of map  $\hat{\Pi}$  for this case are shown in Fig. 1. The properties of the map to large extent are determined by the phase expansion coefficient  $(\partial\phi'/\partial\phi)_h$ . In our case relations (4), (5) imply that

$$\left(\frac{\partial\phi'}{\partial\phi}\right) \sim \varepsilon \left(\frac{h_1}{h}\right)^{5/2}. \quad (6)$$

In the region where  $(\partial\phi'/\partial\phi) \ll 1$ , i.e.  $h < -K_1 h_1 \varepsilon^{2/5}$  with a big enough positive constant  $K_1$ , map  $\hat{\Pi}$  satisfies the assumptions of KAM-theory. According to KAM theory, this region is filled up to a residual set of small measure by invariant curves of map  $\hat{\Pi}$  that enclose cylinder  $\Phi$  (i.e. values of  $\phi$  along any of these curves cover all segment  $[0, 2\pi] \bmod 2\pi$ ) and are close to curves  $h = \text{const}$ . The residual set of small measure is filled up to another residual set of much smaller measure by invariant curves that do not enclose cylinder  $\Phi$  (i.e. values of  $\phi$  along each such curve cover only a part of segment  $[0, 2\pi] \bmod 2\pi$ ). These invariant curves are organized into "islands" corresponding to different resonances. For phase points on invariant curves the motion of the light shell is quasi-periodic (in particular case, periodic). For a phase point not on invariant curves the motion of the light shell can be chaotic. However, because the trajectory of the phase point on  $\Phi$  remains locked between invariant curves, the light shell can not acquire the positive energy and escape to infinity. The invariant curves with largest  $h$  value that can be constructed by KAM theory have  $h \sim -\varepsilon^{2/5}$ , the variation of  $h$  along any such curve is also of the order of  $\varepsilon^{2/5}$ . Fig. 2 represents results of numerical experiment in which for several values of  $\varepsilon$  we looked for the invariant curve with the maximal value of  $h$  enclosing cylinder  $\Phi$ . The least mean square fit gives the scaling  $h \sim -\varepsilon^{0.46}$ .

In the region where  $(\partial\phi'/\partial\phi) \gg 1$  small changes in  $\phi$  lead to large changes in  $\phi'$ . Therefore, values of  $\phi$  and  $\phi'$  can be considered as statistically independent according to phase expansion criterion. This is the region where  $-K_2^{-1} h_1 \varepsilon^{2/5} < h < 0$ , with  $K_2$  being a large enough positive constant. In this region, variations of the light shell energy  $h$  during one iteration of map  $\hat{\Pi}$  can be treated as independent random values. As a result the changes of the energy along the phase point trajectory are of diffusional type. Apparently, as a result of this diffusion almost all phase points in the region under consideration acquire positive energy, i.e. light shell escape to infinity. Numerical experiment (see

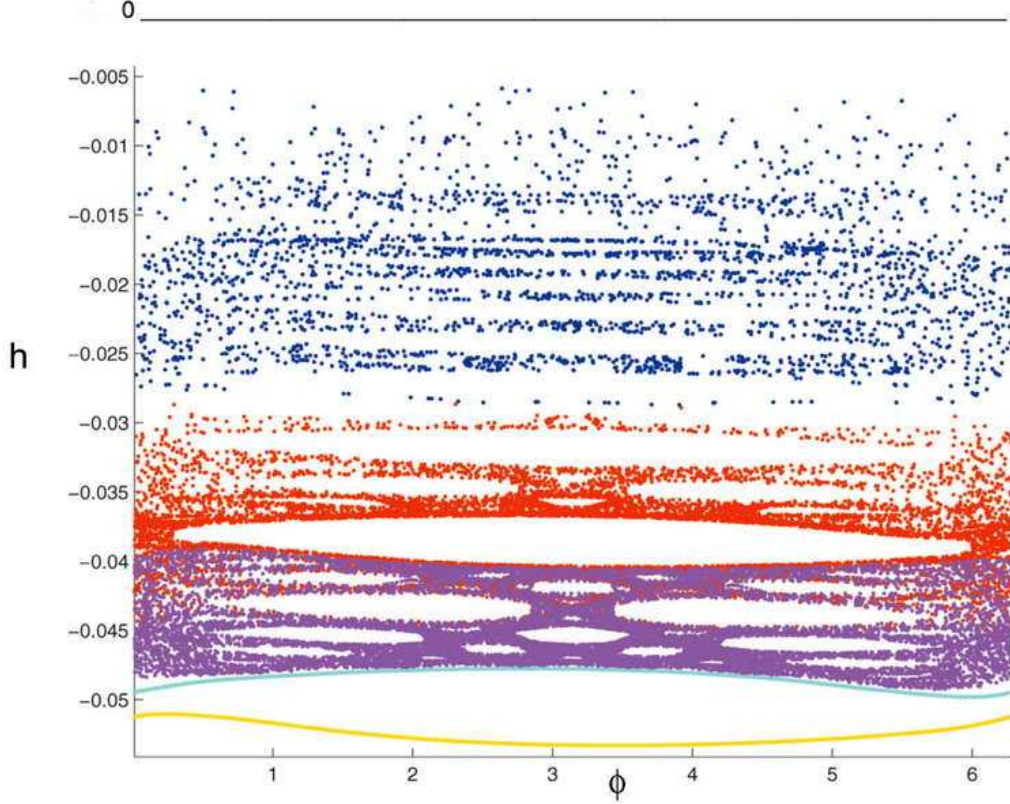


Figure 1: Trajectories of five initial points (3 chaotic, 2 regular) under the action of Poincaré return map in the domain where lines  $h = \text{const}$  are of the type A.  $m_1/M = 0.005$ ,  $J_1 = 1.46\sqrt{2}$ ,  $J_2 = 1.4\sqrt{2}$ .

Fig. 2, straight line 2) gives for the lower boundary of this domain the scaling  $h \sim -\varepsilon^{0.42}$ . The lower boundary of this domain was defined by the condition that there are no visible islands in the numerical experiment above this boundary.

In the intermediate region  $-K_1 h_1 \varepsilon^{2/5} < h < -K_2^{-1} h_1 \varepsilon^{2/5}$ , there are both regular and chaotic trajectories, and apparently the measures of sets of corresponding phase points are of the same order  $\varepsilon^{2/5}$ .

In any annulus  $\gamma \leq h \leq \beta < 0$  with values  $h$  of the type A the dynamics is described by KAM theory (here  $\beta, \gamma$  are constants, i.e. values independent of  $\varepsilon$ ,  $\gamma < \beta$ ). Such an annulus is filled up to residual set of measure  $O(\sqrt{\varepsilon})$  by invariant curves enclosing cylinder  $\Phi$ ; these curves are  $O(\sqrt{\varepsilon})$ -close to circles  $h = \text{const}$ . The residual set of measure  $O(\sqrt{\varepsilon})$  is filled up to residual set of much smaller measure, apparently  $O(\exp(-\text{const}/\sqrt{\varepsilon}))$ , by invariant curves that do not enclose cylinder  $\Phi$  [2]. These curves are organized into resonant "islands". In this annulus, therefore, chaotic motion can oc-

cupy only measure  $O(\exp(-\text{const}/\sqrt{\varepsilon}))$ .

Let us now consider the case when value  $h = 0$  is of the type B. Consider in cylinder  $\Phi$  the annulus  $\alpha \leq h \leq 0$  such that all values  $h$  for this annulus are of type B (here  $\alpha$  is a constant,  $\alpha$  does not depend on  $\varepsilon$ ). In this annulus map  $\hat{\Pi}$  is continuous, but it is not smooth. Singularities of  $\Pi$  are described in Sec. III. KAM-theory is not applicable to such a map. Trajectories of several initial points under the action of  $\hat{\Pi}$  in this case are shown in Fig. 3. Instead of invariant curves, some invariant sets of complicated structure appear, that apparently have non-zero measure. (The structure of one of these sets under magnification is shown in Fig. 4; for similar structure for discontinuous map see [3] and references there). Numerics show that, similarly to the smooth case, in this singular case there are several regions (see Figs. 5 – 6): 1) the region where there are no visible islands; in this region apparently almost all trajectories are chaotic and correspond to escape to infinity; 2) the re-

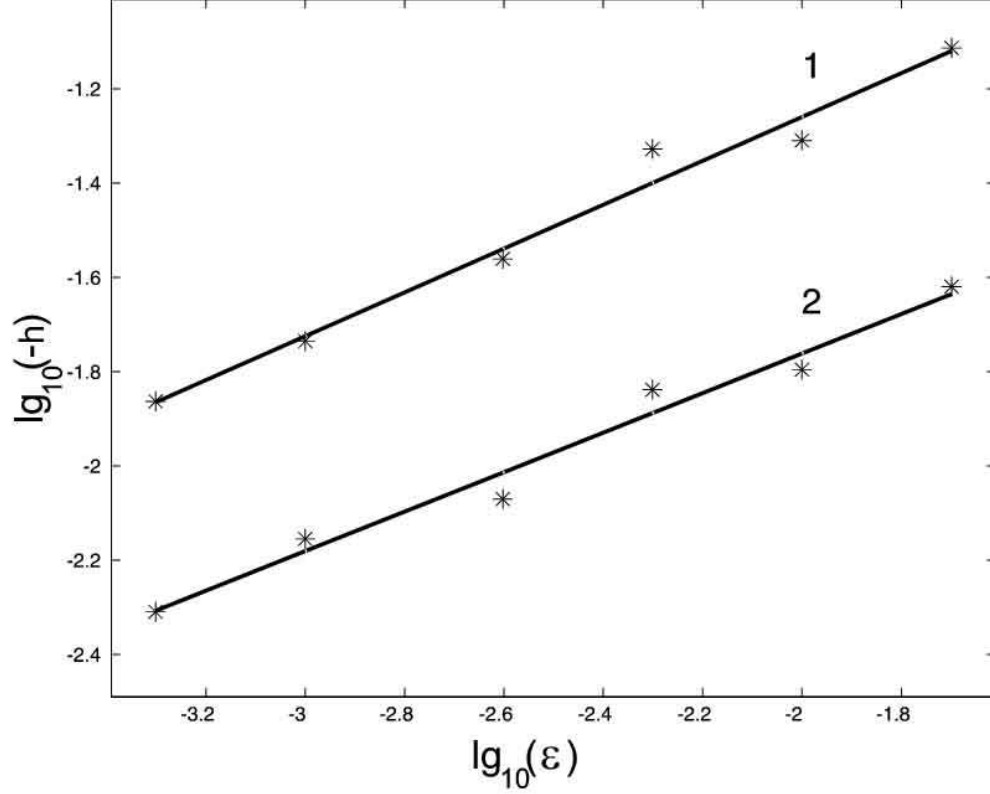


Figure 2: Maximal values of  $h$  for the invariant curves enclosing cylinder  $\Phi$  and minimal value of  $h$  such that there are no visible islands at bigger values of  $h$ . The line  $h = 0$  is of the type A.

gion where there are islands, there are phase points corresponding to quite fast escape to infinity and there are also phase points outside the islands that do not escape for very long time ( $10^6$  iterations of map  $\hat{\Pi}$  do not lead to escape); about these latter phase points it is not clear if they are captured forever or not; 3) the region where there are islands and all phase points outside the islands do not escape for very long time; again about these latter phase points it is not clear if they are captured forever or not.

One of islands inside the domain where lines  $h = \text{const}$  are of the type B is shown in Fig. 7. For all initial points inside such an island the light shell on the period of its motion passes through the heavy shell with non-zero relative velocity and the number of passages of the shells through each other is the same. Therefore, the Poincaré return map on such an island is an analytic map and KAM-theory is applicable for description of dynamics inside this island.

Numeric simulations give for the lower boundary of the domain where there are no visible islands the scaling  $h \sim -\varepsilon^{0.38}$  (see Fig. 5).

## V. DYNAMICS OF MOTION AND POINCARÉ RETURN MAP FOR THE NON RESTRICTED PROBLEM.

In the non restricted problem the motion of shells is described by Hamiltonian  $H = H_1 + H_2$ . In this problem  $H$  is the total energy of the system, and it is an integral of motion. Here

$$H_1 = \frac{p_1^2}{2m_1} + V_1, \\ V_1 = \begin{cases} V_{10}, & \text{if } r_1 \leq r_2, \\ V_{10} - Gm_2m_1/r_1, & \text{if } r_1 \geq r_2, \end{cases} \quad (7)$$

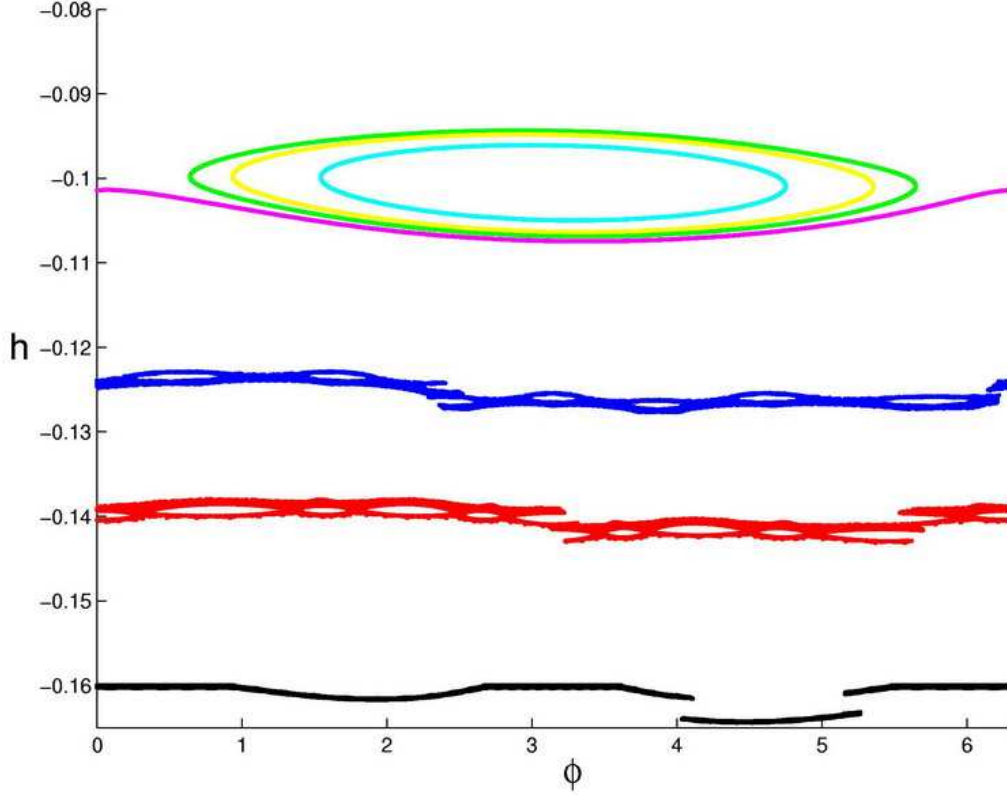


Figure 3: Trajectories of seven initial points under the action of Poincaré return map in the domain where lines  $h = \text{const}$  are of the type B and there are no visible escaping trajectories.  $m_1/M = 0.0003$ ,  $J_1 = 1.4\sqrt{2}$ ,  $J_2 = 1.2\sqrt{2}$ ,  $H_1 = -0.09$ .

where

$$V_{10} = -\frac{G(M + m_1/2)m_1}{r_1} + \frac{J_1^2 m_1}{2r_1^2},$$

and

$$H_2 = \frac{p_2^2}{2m_2} + V_2, \quad V_2 = \begin{cases} V_{20}, & \text{if } r_2 \leq r_1, \\ V_{20} - Gm_1 m_2 / r_2, & \text{if } r_2 \geq r_1, \end{cases} \quad (8)$$

with

$$V_{20} = -\frac{G(M + m_2/2)m_2}{r_2} + \frac{J_2^2 m_2}{2r_2^2}.$$

Here  $p_1$  is the momentum canonically conjugated to radius  $r_1$ , and  $p_2$  is the momentum canonically conjugated to radius  $r_2$ . Therefore, in the nonrestricted problem motion of the shells is described by a Hamiltonian system with two degrees of freedom.

We have constructed Poincaré return map for the non restricted problem. The total energy of system  $H$  is fixed and the shells with numbers "1" and "2" are now in equal rights. We have constructed the Poincaré return map in the same way as in section III. At the time moments  $t$ , when the values of  $r_2$  have local minima on the trajectory, we mark on the cylinder  $\Phi = \{(h, \phi) : -\infty < h < \infty, \phi \bmod 2\pi\}$  the values of the shell "2" energy  $H_2$ , and the shell "1" mean anomaly  $l_1 : (H_2, l_1) = (h, \phi)$ . We present one such trajectory for equal masses  $m_1 = m_2 = 0.007$  in Fig 8. One see here chaotic diffusion and a complicated structure of island.

## VI. CONCLUSION.

The shell approximation is often used for description of spherical stellar clusters dynamics. We



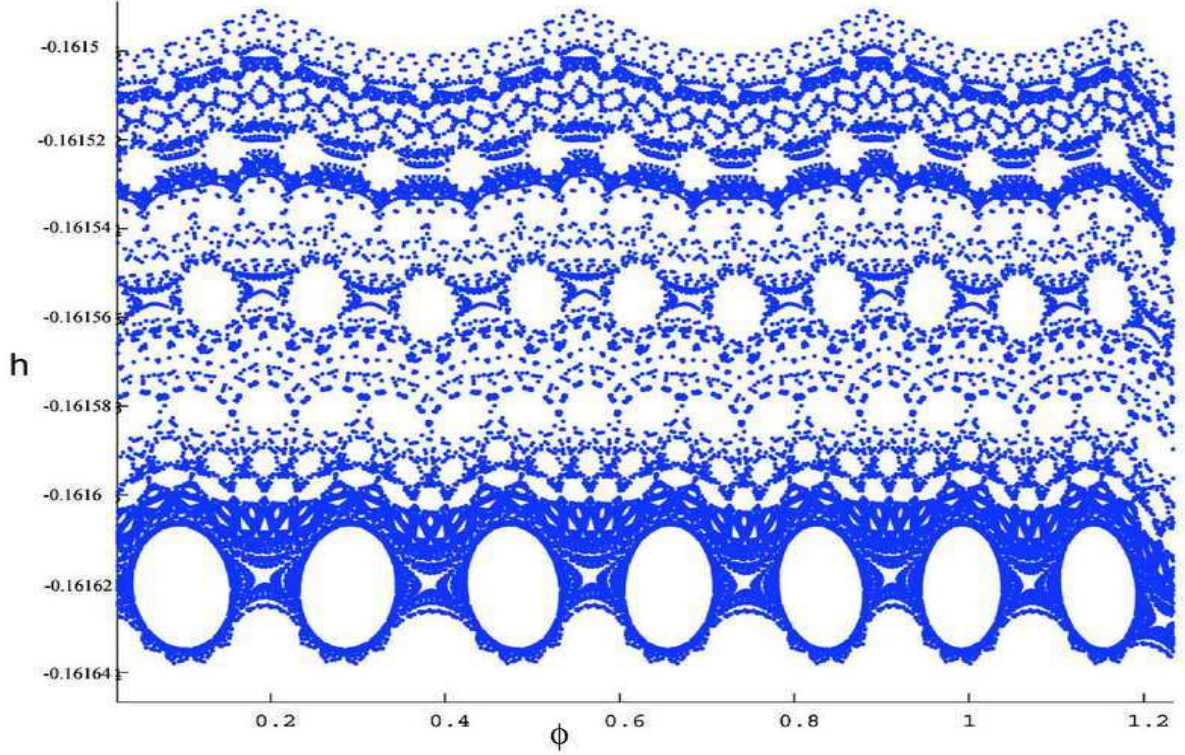


Figure 4: The structure of invariant set from Fig. 3 under magnification.

have considered dynamics of two shells in the case when one can neglect the influence of one ("light") shell onto the motion of another ("heavy") shell. It is demonstrated that dynamics of the light shell at small enough absolute value of negative energy is chaotic and leads to escape of the shell to infinity. There are two types of parameter values for the system. For one type the Poincaré return map is analytic in the domain of small negative energies of the light shell. Therefore, KAM-theory is applicable for the description of the dynamics provided that the mass of the heavy shell is much smaller than the mass of the central body. In this case we have described the structure of the phase space and have obtained the scaling laws for the measure of the domain of chaotic motion and for the minimal

energy of the light shell sufficient for escape to the infinity. For another type of parameter values, the Poincaré return map has singularities in the domain of small negative energies of the light shell. In this case KAM-theory is not applicable and it looks like currently there is no general theory which would allow to deal with this situation. Numerical calculation show coexistence of invariant sets of complicated structure and chaotic trajectories with final escape to infinity.

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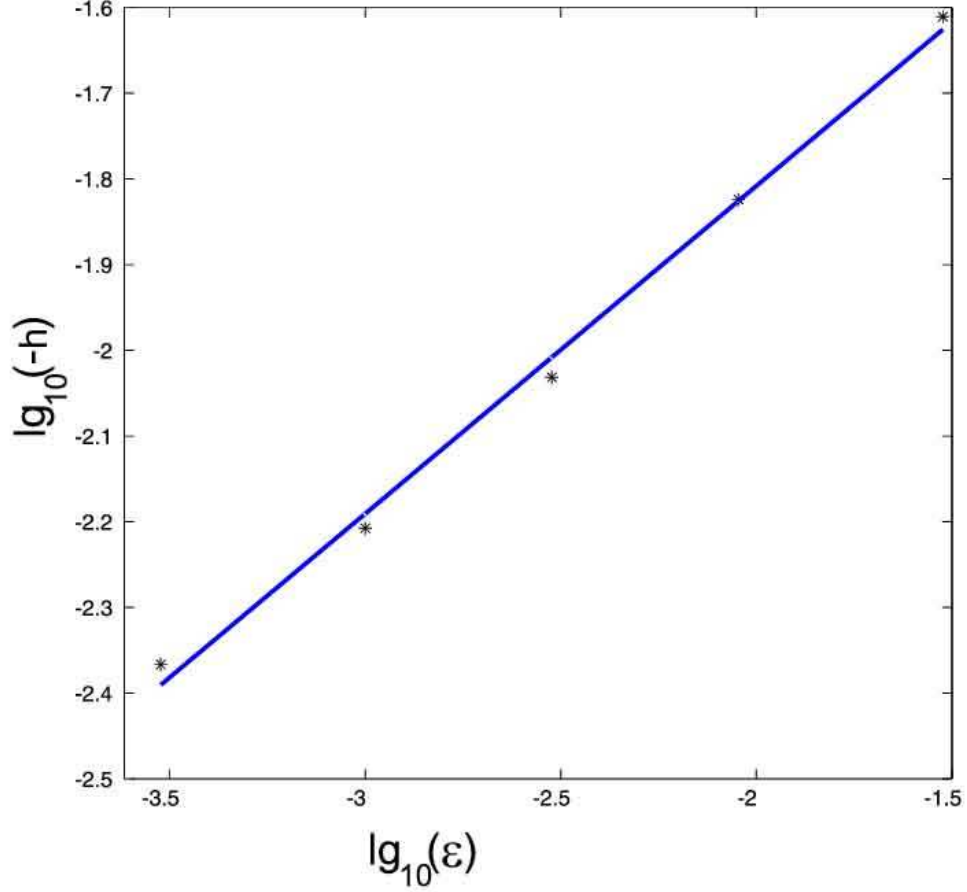


Figure 5: Minimal values of  $h$  such that there are no visible islands at bigger values of  $h$ . The line  $h = 0$  is of type B.

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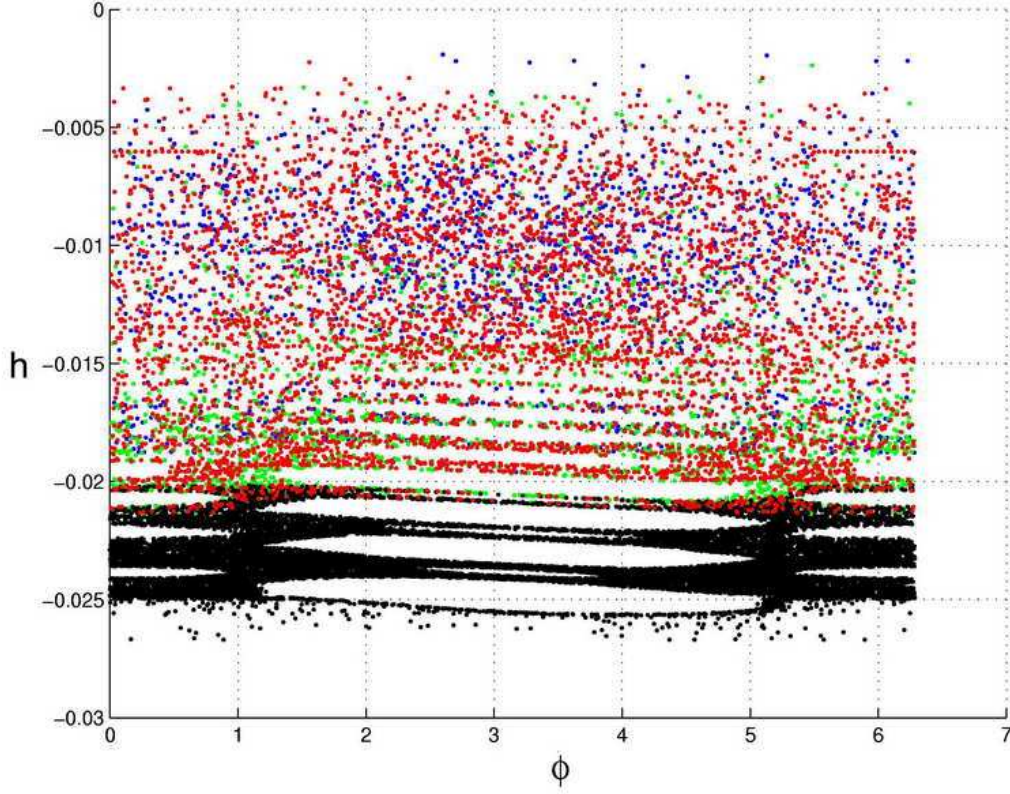


Figure 6: Trajectories of four initial points under the action of Poincaré return map for the case when the line  $h = 0$  is of type B.  $m_1/M = 0.01$ ,  $J_1 = 1.5\sqrt{2}$ ,  $J_2 = 2.0\sqrt{2}$ ,  $H_1 = -0.09$ .

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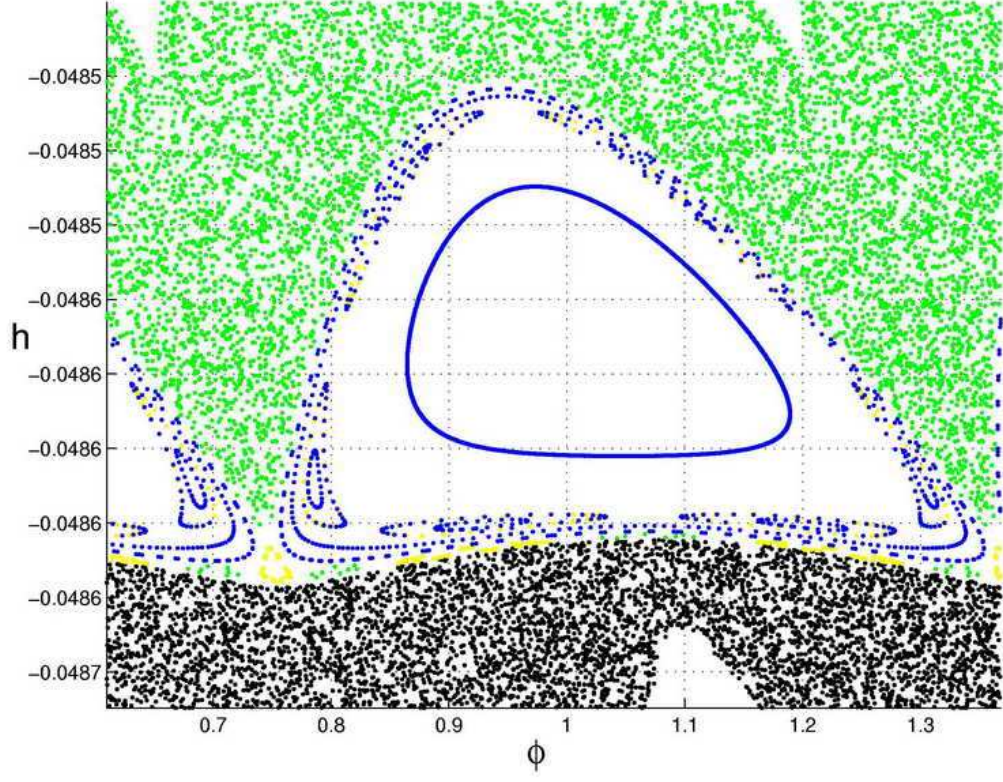


Figure 7: Stability island inside the domain where lines  $h = \text{const}$  are of type B.

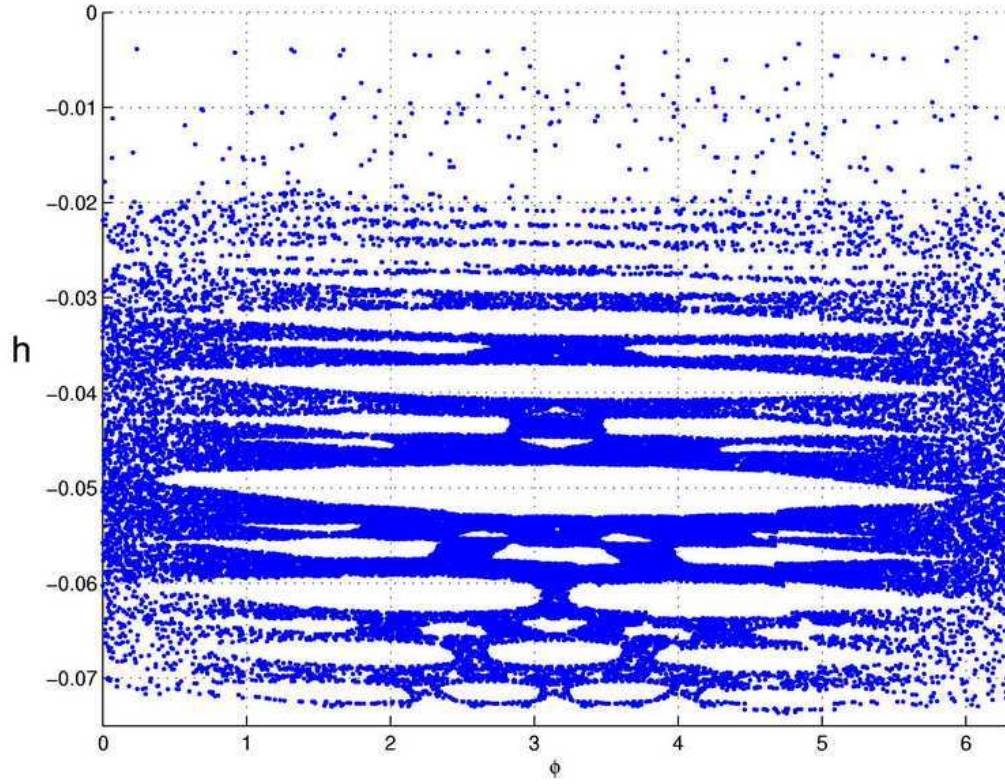


Figure 8: Trajectory of one initial point under the action of Poincaré return map for the nonrestricted problem. Parameters and initial conditions are:  $m_1 = m_2 = 0.007$ ,  $J_1 = 1.5\sqrt{2}$ ,  $J_2 = 1.51\sqrt{2}$ ,  $H_1/m_1 = -0.05$ ,  $H_2/m_2 = -0.05$ ,  $r_1 = 5.1$ ,  $r_2 = 5.0$ , the initial direction of the shells motion is inwards.